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INTEGRAL CONSTRAINTS ON THE CONTROLS IN AN ENCOUNTER GAME*

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The players /1/ have, in three, motors of different construction with noninterchangeable fuels. The first motors have, in mathematical notation, "geometric" constraints, the second, "power-integral" constraints, and the third, impulse constraints. The payoff is a prescribed function of position at a fixed termination time. The problem includes as a special cases the game of one motor of any type against any of the three types.

It is assumed that the problem's singular sets are concentrated in a plane relative to which the game has symmetry. Functions of two finite-dimensional vectors and the maximum with respect to the second vector's components of the minimum of the function with respect to the first vector's components have been constructed. Conditions have been ascertained under which the maximum constructed is the game's value. In the first example two points in a central field play on the absolute value of the difference of polar angles. In the second, a generalization of the Littlewood problem, a man on the boundary of a circular arena plays with a lion (the payoff is distance). In eight subcases the value and the best controls have been constructed for all positions.

1. By $\rho^{(k)}$ we denote a k-dimensional space: the integers $i = 1, 2,\ j = 1, 2, 3$ (i is the player's number). We introduce the vectors and numbers

$$z_i \in \rho^{(2)},\ \mu_i = (\mu_{i,1},\ \mu_{i,2},\ \ \mu_{i,3}),\ \mu_{i,j} \in \rho^{(1)}$$
$$\nu_i = (\nu_{i,1},\ \nu_{i,2},\ \ \nu_{i,3}),\ \ \nu_{i,j} \in \rho^{(1)},\ \varepsilon_{i,j} = |\nu_{i,j}|,\ \ \tau \in \rho^{(1)}$$
$$u_i = (u_{i,1},\ u_{i,2},\ u_{i,3}),\ u_{i,j} \in \rho^{(2)}$$
$$x^{(i)} = (z_i,\ \mu_i,\ \nu_i,\ \tau),\ x = (x^{(1)},\ x^{(2)}),\ u = (u_1,\ u_2)$$

The vectors z_i are the players' geometric coordinates, the numbers $u_{i,j}$ are the resources of the players' controls $\mu_{i,j}$. The auxiliary sets

$$U_{\varepsilon,i}(x) = \{u_i \mid |u_{i,j}| \leqslant \varepsilon_{i,j}{}^{1-j} |\mu_{i,j}| \text{ for } j = 1, 2, 3\};$$
$$U_\varepsilon = U_{\varepsilon,1} \times U_{\varepsilon,2}$$

form the basic sets

$$U_i = (\bigcup U_{\varepsilon,i} \text{ along } \nu_i) \ni u_i,\ u \in U = U_1 \times U_2$$

We specify the continuous second-order square diagnonal matrices $A_i(\tau)$ and we write out the equations of motion

$$z_i^{\cdot} = \varphi_{z,i}(z_i, \tau, u_{i,1},\ u_{i,2}) + A_i(\tau) u_{i,3}$$
$$\mu_{i,1}^{\cdot} = \mu_{i,2}^{\cdot} + u_{i,2}^2 = \mu^{\cdot}_{i,3} + |u_{i,3}| = 0,\ \tau^{\cdot} + 1 = \nu_i^{\cdot} = 0$$

We let the functions $\varphi_{z,i}(x, u)$ have continuous partial derivatives. The adjoint vector

$$p = (p_1, p_2),\ p_i = (p_{z,i},\ p_{\mu,i},\ p_{\nu,i},\ p_\tau)$$

has the dimension of vector x. By $\{x\}$ we denote the space of positions, $x_\tau = (z_1, \mu_1, z_2, \mu_2)$,and we consider the vectors $x_1 \in \{x\}$, $z_0 = (x_1, x)$. The functions

$$u_\xi(z_0) = (u_{\xi,1}(z_0),\ \ u_{\xi,2}(z_0)) \in U_\varepsilon(x)$$

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are continuous in x_τ and measurable in τ ; also $u_{\xi,i}(z_0) \in U_{\varepsilon,i}(x)$. We combine them into the sets $U_{\xi,i}$, U_ξ. We introduce the sets $\xi_i(x_1) \in \{x\}$ and, in addition,

$$\xi_\varepsilon(x_1) = \{x \mid |x - x_1| - \varepsilon < 0\} \subset \xi_i(x_1)$$

We construct the sets

$$\text{for } \varepsilon = \varepsilon_1 + \varepsilon_2;\ \varepsilon_i = |\nu_i|;\ \xi_i(x) \in V_{\xi,i}$$

$$v_i = \{u_{\xi,i}(z_0),\ \xi_i(x_1)\},\ w_i = \{u_\xi(z_0),\ \xi_i(x_1)\}$$
$$v_{1,i} = \{x_1, v_i\},\ w_{1,i} = \{x_1, w_i\},\ V_i = U_{\xi,i} \times V_{\xi,i}$$

We prescribe

$$w_{1,i},\ e_{w,i}(x_1, x) = e(x, u_\xi(z_0))$$

and we construct the motion $x_v(t)$ of set $w_{1,i}$, satisfying the following equations. The sequence

$$(t_j, x_j = x_v(t_j)), x_1 = x_v(t_1), t_1 = 0$$

corresponds to the equalities

$$t_{j+1} = \inf \{t \mid (t > t_j,\ x_v(t) \overline{\in} \xi_i(x_j)) \vee (t > t_j + 1)\}$$

while the absolutely continuous function $x_v(t)$ corresponds for almost all t corresponds to the equation

$$x_v^{\cdot}(t) = e_{w,i}(x_j, x_v(t)) \text{ for } t \in [t_j, t_{j+1}]$$

The motion exists for any set $w_{1,i}$. We specify the payoff $r(x)$ and we compute

$$X_v(v_{1,i}) = \{x_v(\tau) \mid u_{\xi,j} \in U_{\xi,j},\quad j \neq i\}$$

$$r_v(v_{1,i}) = (-1)^{i+1} (\sup (-1)^{i+1} r(x_v(\tau)) \text{ along } x_v(\tau) \in X_v(v_{1,i}))$$

$$b_i(x_1) = (-1)^{i+1} (\inf r_v(v_{1,i}) \text{ along } (v_i \in V_i,\ \varepsilon > 0))$$

$$\lim r_v(\{x_1, v_{0,i}(z_0)\}) \text{ as } \varepsilon \to 0 = b_i(x_1)$$

($b_1(x)$ is the first player's game value $v_{0,1}(z_0)$ is his best strategy). The problem is posed analogously for the second player.

2. The letters f, $\mathbf{w}$, v, w, γ, δ, ξ, η, ζ, σ, θ and capital letters will be used to denote sets. If a letter in a list has a letter subscript, and there is a figure one after the subscript, then this is a scalar or a vectorial function. We specify the set

$$w_\alpha = \{w_{\alpha,1}, w_{\alpha,2}\}$$
$$w_{\alpha,1} = c(y, u) \in \rho^{(1)};\ w_{\alpha,2} = w^{(j)}_{\alpha,2,1} \times w^{(j)}_{\alpha,2,2},\ w^{(j)}_{\alpha,2,i} \in U_i(x)$$
$$w^{j)}_{\alpha,2,j} = \{u_j \mid u \in w_{\alpha,2}\}$$
$$w^{(j)}_{\alpha,2,i} = \{u_i \mid w^{(j)}_{\alpha,2,j}(y, u_i) \neq \varnothing\}$$

We compute the operators $f_1(w_\alpha)$ (minimum) $f_2(w_\alpha)$ (maximum)

$$f_i(w_\alpha) = \{w_{i,\alpha,1},\ w_{i,\alpha,2}\} = w_{i,\alpha}$$
$$w_{i,\alpha,2} = w^{(i)}_{i,\alpha,2,1} \times w^{(i)}_{i,\alpha,2,2}$$
$$w_{i,\alpha,1} = (-1)^{i+1} \inf (-1)^{i+1} (w_{\alpha,1}(y, u) \text{ along } u_i \in w^{(i)}_{\alpha,2,i}(y, u_j))$$
$$w^{(i)}_{i,\alpha,2,i} = \{u_i \mid w_{i,\alpha,1}(y, u_j) - w_{\alpha,1}(y, u) = 0,\ u \in w_{\alpha,2}(y)\}$$
$$w^{(i)}_{i,\alpha,2,j} = w^{(i)}_{\alpha,2,j}(y)$$

The operators $f_i(w_\alpha)$ for $w_{\alpha,2} \in \{p\}$ (the set $\varphi(x) \ni p$ is defined below) and the operators $f_{i,t}(w_\alpha) = f_i(w_\alpha)$ for $w_{\alpha,2} \subset \theta_\tau = \{t \mid t \geqslant 0,\ t \in [0, \tau]\}$ are defined by analogy. The operators $f_{i,j}(w_\alpha) = f_i(f_j(w_\alpha))$. To the functions

$$c(y) \in p^{(1)}\ h = pe(x, u)$$
$$h_t = \partial h(y, u) \mid \partial y^0,\ y = (p, x),\ y^0 = (-x, p)$$

correspond the operator

$$f_u(c(y)) = (\lim (t^{-1}(c(y + h_t t) - c(y))) \text{ as } t \to +0) = \beta_c(y, u)$$

We write the sets

$$X = \{x \mid \mu_{i,j} \geqslant 0 \text{ for } i = 1, 2, j = 1, 2, 3\}$$
$$\mu_{i,\delta} = p_{\xi,i} = (p_{\nu,i,1},\ p_{\nu,i,2})$$
$$P_i = \{p_i \mid |p_{z,i}| > 0,\ (p_{\mu,i,j}(-1)^j > 0$$
$$\text{for } j = 1, 2, 3),\ |p_{\xi,i}| \leqslant \mu_{i,3}\}$$
$$\varphi(x) = P = P_1 \times P_2,\ \gamma = X \times P \ni y,$$
$$p_\xi = (p_{\xi,1}, p_{\xi,2})$$

Let
$$\gamma_c = \{y \mid p_\xi = 0\} \cap \gamma,\ w_c = f_{2,1}(\{h(y, u),\ U_\varepsilon(x)\})$$
We assume that the equation
$$y^{\cdot} = e_c(y) = \{\partial h(y, u)/\partial y^0 \mid u \in w_{c,2}(y)\}$$
admits, for $y_c(y, 0) = y \in \gamma_c$, of a unique absolutely continuous solution $y_c(y, t)$. We compute
$$y_{c,1}(y, t) = y_{c,0}(y, t) \text{ for } \mu_{i,3} = p_{\xi,i}$$
$$y_{c,0}(y, t) = \lim y_{c,1}(y, t) \text{ as } \varepsilon \to 0$$
We use the notation
$$a_x(y) = x,\ a_{z,i}(y) = a_{z,i}(x) = z_i$$
$$a_{\mu,i,j}(x) = a_{\mu,i,j}(y) = \mu_{i,j},\ a_\tau(x) = \tau$$
$$a_{p,z,i}(y) = p_{z,i},\quad a_{p,\mu,i,j}(y) = p_{\mu,i,j},\quad a_\tau(y) = \tau$$

etc. and in this notation we prescribe the vector $y_b(y, t)$ by the equalities
$$a_{\varepsilon,i,j}(y_b) = \varepsilon_{i,j},\ a_{\mu,i,3}(y_b) = \mu_{i,3} - |p_{\xi,i}| + a_{\mu,i,3}(y_{c,1}(y, t))$$
$$a_0(y_b) = a_0(y_{c,1}(y, t))$$
The notation $a_0(y_b) = a_0(y_{c,1})$ signifies the coincidence of the remaining components of vectors $y_b, y_{c,1}$. The motions $y_b(y, t)$ are the foundation for the construction and have two or fewer jumps. We construct the sets
$$U_{i,\delta} = \{u_{i,3} \mid |u_{i,3}| = |p_{\xi,i}|,\ u \in w_{c,2}(y)\}$$
$$U_\delta(y) = U_{1,\delta} \times U_{2,\delta}$$
$$Y_{\delta,i} = \{y \mid |p_{\xi,i}| > 0,\ U_{i,\delta}(y) \neq \varnothing\},\ Y_\delta = Y_{\delta,1} \cup Y_{\delta,1}$$
The set
$$v_\alpha = \{c_\alpha(y),\ y_\alpha(y, t),\ \gamma_\alpha\}$$
consists of the function $c_\alpha(y) \in \rho^{(1)}$ of the vector $y_\alpha(y, t) \in \gamma$ and the set $\gamma_\alpha \subset \gamma$. We construct
$$\theta_\alpha(v_\alpha) = \{t \mid y_\alpha(y, t) \overline{\in} \gamma_\alpha\},\ t_a(v_\alpha) = \inf \theta_\alpha(v_\alpha)$$
$$y_a(v_\alpha) = y_\alpha(y, t_a(v_\alpha)),\ c_a(v_\alpha) = c_\alpha(y_a(v_\alpha))$$
$$\theta_a(v_\alpha) = \{t \mid t \in [0, t_a(v_\alpha))\}$$
If $\theta_\alpha = \Phi$, then $t_a = c_a = \infty$.

Now for the set
$$v_t = \{0,\ y_{c,1}(y, t),\ \gamma_t \setminus Y_\delta\}$$
we compute $t_1(y) = t_a(v_t)$, $y_1(y) = y_a(v_t)$. We construct the vector $y_{\delta,i}(y)$, i.e., the result of the jump $y \to y_{\delta,i}(y)$
$$a_{z,i}(y_{\delta,i}) = z_i + A_i(\tau)\, p_{\xi,i},\ a_{\mu,i,3}(y_{\delta,i}) = \mu_{i,3} - |p_{\xi,i}|,\ a_{p,\xi,i}(y_{\delta,i}) = 0$$
$$a_0(y_{\delta,i}) = a_0(y)$$
$$y_{j,b}(y, t) = \lim y_b(y, t + t_1^0) \quad \text{as} \quad (-1)^j t_1^0 \to +0$$
Let $y_1(y) \in Y_{\delta,i} \setminus Y_{\delta,j}$, then $t_1(y) = t_{\delta,i}(y)$, while the jump
$$y_1 \to y_{\nu,i}(y) = y_{\delta,i}(y_1) \text{ for } p_{\xi,i} = u_{\delta,i}(y_1)$$
The function $u_{\delta,i}(y_1) \in U_{\delta,i}(y_1)$ is unique. The next jump is constructed by analogy. Such is the description of the motions $y_b(y, t)$. We write them out in greater detail
$$y_b(y, t) = y_{2,b}(y, t) = y_{1,b}(y, t) \text{ for } t \in [0, t_1(y))$$
$$y_b(y, t_1(y) + t) = y_b(y_{\nu,i}(y_1(y)), t)$$
$$\text{for } t \in [0, t_1(y_1(y)) \quad \text{etc.}$$
We describe the game's symmetry property by denoting $z = z_{2,1} - z_{1,1}$, $a_z(y) = z$, $y_2(y)$ is a symmetrical vector
$$a_{z,i,1}(y_2) = -z_{i,1},\ a_{p,z,i,1}(y_2) = -p_{z,i,1};\ a_0(y_2) = a_0(y)$$
We assume that the payoff function $r(x) \geqslant 0$ is convex in z and corresponding to the symmetry equation
$$r(a_x(y_b(y, t))) = r(a_x(y_b(y_2(y), t)))$$
for $t \in \theta_\tau(x)$.

We shall reckon that the set $\xi^0 = \{x \mid b_1(x) = b_2(x) = 0\}$ is known and we shall carry the construction out for $x \in \xi = X \setminus \xi^0$. Let $\delta \in X$. We denote $\gamma^0(\delta) = \{y \mid x \in \delta\} \cap \gamma$. We consider the series of sets and operators

$$v_\alpha = \{c_\alpha(y),\ y_\alpha(y,t),\ \gamma_\alpha\}$$
$$\varphi_\alpha(x) = \{p \mid y \in \gamma_\alpha\} = \varphi_\alpha(v_\alpha)$$
$$\zeta_\alpha(v_\alpha) = f_{2,1}(\{c_\alpha(y),\ \varphi_\alpha(x)\})$$
$$\alpha_\xi(x) = \alpha_\xi(y) = \alpha_\xi(v_\alpha) = \zeta_{\alpha,1}(v_\alpha)$$
$$\zeta_\beta(v_\alpha) = f_{2,t}(\{\alpha_\xi(y_b(y,t)),\ \theta_a(y)\})$$
$$\alpha_\beta(y) = \zeta_{\beta,1}(v_\alpha)$$
$$\zeta_\gamma(v_\alpha) = f_{2,1}(\{\alpha_\beta(y),\ \varphi_\alpha(x)\})$$
$$\xi_\zeta(v_\alpha) = \{x \mid \zeta_{\alpha,1}(x) = \zeta_{\gamma,1}(x)\}$$
$$\xi_\xi(v_\alpha) = \{x \mid y \in \gamma_\alpha,\ x \overline{\in} \xi_\zeta(v_\alpha)\}$$

These operators will be used below. We begin with the set

$$w_{1,\beta} = \{r(x),\ y_b(y,t),\ \gamma^0(\xi)\}$$
$$\gamma^0(\xi) = \{y \mid \tau \geqslant 0,\ x \overline{\in} \xi^0\}$$

and we compute the programmed maximum $\alpha_{1,\xi}(x)$ /2/ and the operators

$$\alpha_\xi(\mathbf{w}_{1,\beta}) = \alpha_{1,\xi}(x),\ \xi_\zeta(\mathbf{w}_{1,\beta}),\ \xi_\xi(w_{1,\beta})$$
$$h_\alpha(y) = \mathbf{w}_{\gamma,1}(y),\ \mathbf{w}_\gamma(y) = f_{1,t}(\{a_z(y_b(y,t)),\ \theta_a(\mathbf{w}_{1,\beta})\})$$

We consider the set $\xi_\alpha = \xi_\xi(\mathbf{w}_{1,\beta}) \cap \{x \mid z \geqslant 0\}$ and we compute

$$\zeta_h = f_{2,1}(\{-h_\alpha(y),\ \zeta_{\alpha,2}(\mathbf{w}_{1,\beta})\})$$
$$\sigma_{1,h}(x) = \zeta_{h,1}(\mathbf{w}_{1,\beta})$$

In many problems the set

$$\xi_\alpha = \{x \mid x \in \xi_\alpha,\ \ \sigma_{1,h}(x) < 0\}$$

Let

$$w_{1,\gamma} = \{\alpha_{1,\xi}(x),\ y_b(y,t),\ \gamma_1{}^0(\xi_\alpha)\};\ \ \gamma_1{}^0(\xi_\alpha) \in \gamma^0(\xi_\alpha)$$

The set $\gamma_1{}^0 = \gamma_{1,\xi} \cup \gamma_{2,\xi}$ consists of two sets

$$\gamma_{1,\xi} = \{y \mid h_\alpha(y) = 0,\ x \in \xi_\alpha\}$$
$$\gamma_{2,\xi} = \{y \mid |p_{\xi,1}| = |p_{\mu,1,2}|^{-1} = 0,\ h_\alpha(y) < 0\}$$

It can be shown that the set

$$\varphi_h(x) = \{p \mid y \in \gamma_1{}^0(\xi_\alpha)\} \neq \varnothing \text{ for } x \in \xi_\alpha$$

We construct the sliding function $t_\lambda(y)$ and the sliding set θ_λ

$$t_\lambda(y) = \inf\{t \mid t \in \theta_a(\mathbf{w}_{1,\beta}),\ a_z(y_b(y,t)) < 0\} \text{ for } y \in \gamma_{2,\xi}$$
$$t_\lambda(y) = \sup\{t \mid t \in \theta_a(\mathbf{w}_{1,\beta}),\ a_z(y_b(y,t)) = 0\} \text{ for } y \in \gamma_{1,\xi}$$
$$\theta_\lambda = \{y \mid t_\lambda(y) = 0,\ y \in \gamma_1{}^0(\xi_\alpha)\}$$

The set θ_λ yields the motion

$$y_\rho(y,0) = y \in \theta_\lambda,\ y_\rho(y,t) \in \theta_\lambda$$

We describe these motions by forming the functions and sets

$$\lambda = p_{\nu,1,3},\ \beta_z(x,u) = f_u(z)$$
$$h_1(y,u) = h(y,u) + \beta_z(x,u)$$
$$U_h = U(x) \cap \{u \mid \beta_z(x,u) = 0,\ |u_{i,1}| \leqslant \mu_{i,1}\}$$
$$w_h = f_{2,1}(\{h_1(y,u),\ U(x)\})$$
$$e_h(y) = \{\partial h_1(y,u)/\partial y^0 \mid u \in \mathbf{w}_{h,2}(y)\}$$

To determine $\lambda = \lambda_1$ we compose the set

$$\theta_\beta(y) = \{y_1 \mid a_\lambda(y_1) = \lambda_1,\ \ a_0(y_1) = a_0(y),\ w_{h,2}(y_1) \in U_h(x)\}$$

By ρ^1 we denote the collection of single-element sets and we let $\rho^0 = \{z_1 \mid z_1 = 0\}$. We assume that the motion $y_\rho(y,t)$ begins with a jump

$$y \to y_\beta(y) = \theta_\beta(y) \subset \rho^1$$

and henceforth corresponds to the equation $y^\cdot = e_h(y)$. We denote

$$\delta_1 = \gamma_1{}^0(\xi_\alpha) \setminus \theta_\lambda,\ \delta_2 = \theta_\lambda,\ \ y_{\theta,1} = y_b(y,t)$$
$$y_{\theta,2} = y_\rho(y,t),\ \ d_{\theta,1} = d_{0,2} = d(w_{1,\beta}) = d_{\theta,3}$$

three sets and we "glue"

$$w_{\theta,i} = \{d_{\theta,i}(x),\ y_{\theta,i}(y,t),\ \delta_i\}$$
$$\delta_3 = \delta_1 \cup \delta_2;\ i = 1, 2, 3$$

into the operators $w_{3,\theta} = f_c(w_{2,\theta},\ w_{1,\theta})$ for $y_{\theta,3}(y,t) = y_{\beta,2}(y,t)$, and we as well "glue" the motions $y_{\theta,i}$, $i = 2, 1$, obtaining the motion $y_{\theta,3}(y,t)$. The set

$$w_{3,\theta} = \{d_{\theta,3}(x),\ y_{\theta,3}(y,t),\ \delta_3\}$$

and the motion $y_{\theta,3}(y,t)$ are specified by the equalities

$$t_{a,i}(y) = t_a(w_{i,\theta})$$
$$y_3 = \lim y_{\theta,3}(y, t - t_1^0) \text{ as } t_1^0 \to +0$$
$$y_{3,\theta}(y, t_1^0 + t) = y_{i,\theta}(y_3, t)$$
$$\text{for } y \in \delta_i,\ t \in [0, t_{a,i}(y_3)),\ i = 1, 2$$

These equalities uniquely determine the motion $y_{\beta,2} = y_{\theta,3}(y,t)$. To the set

$$w_{2,\beta} = w_{\theta,3} = \{\alpha(\mathbf{w}_{1,\beta}),\ y_{\beta,2}(y,t),\ \gamma_1^0(\xi_\alpha)\}$$

corresponds the function

$$\alpha_a(x) = 0,\ x \in \xi^0$$
$$\alpha_a(x) = \alpha_{1,\xi}(x),\ x \in \xi_\zeta(\mathbf{w}_{1,\beta})$$
$$\alpha_a(x) = \alpha_a(x_2(x)) = \alpha_\xi(w_{2,\beta})$$
$$x \in \xi_\xi(\mathbf{w}_{1,\beta})$$
$$x_2(x) = a_x(y_2(x))$$

In a number of cases it turns out that the function $\alpha_a(x)$ is a solution of the problem.

3. We construct a verification scheme. We denote

$$x_\delta(y) = a_x(y_{\delta,2}(y_{\delta,1}(y))),\ \beta_a(x,u) = f_u(\alpha_a(x))$$
$$w_{i,\Delta} = f_{i,j}(\{\beta_a(x,u),\ U(x)\})$$
$$A_{\Delta,i} = \{x \mid \mathbf{w}_{i,\Delta,1}(x)(-1)^{i+1} > 0\}$$
$$\varphi_\delta = \varphi_{\delta,1} \times \varphi_{\delta,2},\ \varphi_{\delta,i} = \{p_{\xi,i} \mid |p_{\xi,i}| \leqslant \mu_{i,3}\}$$
$$w_{i,\delta} = f_{i,j}(\{\alpha_a(x_\delta(y)),\ \varphi_\delta(x)\})$$

Let $\xi_a \in \xi$ and let the function $\rho_a(x, \xi_a)$ be the distance of point x to the set $\bar{\xi}_a$. We write out the sets and functions

$$r_{\Delta,i}(x) = \rho_a(x, A_{\Delta,i}),\ X_{\delta,i} = \{x \mid \mathbf{w}_{i,\delta,2,i} \ni \rho^0 = \{z_1 \mid z_1 = 0\}\}$$
$$X_{\delta,i} = \{x \mid r_{\delta,i}(x) = \rho_a(x, \xi \setminus X_{\delta,i}) > \varepsilon\}$$
$$U_{\rho,i} = U(x) \text{ for } x \,\overline{\in}\, A_{\Delta,i}$$
$$U_{\rho,i} = \{u \mid f_u(r_{\Delta,i}(x)) = 0\} \cap U(x),\ x \in A_{\Delta,i}$$

The set

$$U_{\rho,i} = U_{1,\rho,i} \times U_{2,\rho,i}$$
$$(U_{j,\rho,i} = \{u_j \mid u \in U_{\rho,i}(x)\}$$
$$U_{i,\rho,i} = \{u_i \mid U_{j,\rho,i}(x, u_i) \neq \varnothing\})$$

enables us to compute

$$w_{\sigma,i} = f_{i,j}(\{\beta_a(x,u),\ U_{\rho,i}(x)\})$$

The function $u_{i,\omega}(x) \in U_{i,\omega}(x)$, where

$$U_{i,\omega} = \{u_i \mid (|u_{i,1}\varepsilon| = 1,\ x \in X_{\Delta,i}) \vee (u \in U(x) \text{ for } x \,\overline{\in}\, X_{\Delta,i})\} \cap \{u_i \mid u \in \mathbf{w}_{i,\delta,2}(x)\}$$
$$u_{i,\omega,1}(x) = \{u_{i,1} \mid u_i = u_{i,\omega}(x)\}$$

We assume that the function $u_{i,\omega}(x)$ is continuous for $x \,\overline{\in}\, A_{\Delta,i}$, while the absolute value of the function $u_{i,\omega,1}(x)$ is continuous for all x. We consider the sets $B_{\Delta,i} = \{x \mid r_{\Delta,i}(x) - \varepsilon < 0\}$ and the vector $x_{1,j}$ specified by the equalities

$$a_{\varepsilon,i,k}(x_{1,j}) = \varepsilon_{i,k} + (\varepsilon_{i,k}/2)(-1)^j,\ k = 1, 2, 3$$
$$a_0(x_{1,j}) = a_0(x)$$

We write out the strategy-sets

$$v_i^0(z_0) = \{u_{\xi,i}^0(z_0),\ \xi_i^0(z_0)\}$$
$$\xi_i^0(x_1) = \{x \mid x_{1,1}(x) \overline{\in} \bar{B}_{\Delta,i}\} \quad \text{for} \quad x_1 \overline{\in} \bar{B}_{\Delta,i}$$
$$\xi_i^0(x_1) = \{x \| x - x_1 | - \varepsilon < 0,\quad x_{1,2}(x) \in B_{\Delta,i}\},\quad x_1 \in B_{\Delta,i}$$

The function $u_{\xi,i}^0(z_0)$ has been defined for $x \in \xi_i^0(x_1)$ by the formulas

$$u_{\xi,i}^0(z_0) = u_{i,\omega}(x_{1,1}(x)),\ x_1 \overline{\in} B_{\Delta,i}$$
$$u_{\xi,i}^0(z_0) = u_{i,\omega}(x_{1,2}(x)),\ x_1 \in B_{\Delta,i}$$

We compute

$$\beta_{d,i}(x) = \mathbf{w}_{i,\sigma,1}(x),\ x \overline{\in} X_{\delta,i}$$
$$\beta_{d,i}(x) = \mathbf{w}_{i,\delta,1}(x),\ x \in X_{\delta,i}$$
$$D = \{x \mid \beta_{d,1}(x) = \beta_{d,2}(x) = 0\}$$

We investigate the set $w_i^0 = w_i$ for $v_i = v_i^0(z_0)$. We denote by $C_{v,i}(x_1)$ the admissibility set of player j for $v_i = v_i^0(z_0)$ and we construct the set

$$w_\omega = \{\alpha_a(x),\ x_v\,[w_i^0, t],\ \{x \mid \tau - \tau_1 + \sqrt{\varepsilon} > 0\}\},\quad x_v\,[w_i^0, t] \in X_v(\{x_1, v_i^0(z_0)\})$$

We can show the existence of the function $C_i(x) > 0$ bounded for $x \in C_{v,i}(x_1)$ and corresponding to the estimate

$$(-1)^{i+1}(\alpha_a(x_a(w_\omega)) - \alpha_a(x_1)) \leqslant C_i(x_1)\sqrt{\varepsilon}$$

This estimate permits us to assert that the equalities

$$b_1(x) = b_2(x) = \alpha_a(x);\quad v_{0,i}(z_0) = v_0^i(z_0),\quad z_0 \in D_a \times D_a$$

are valid in a set $D_a \in \xi$ for which the inclusion $x_v\,[w_i^0, t] \in D_a$ follows from the inclusion $x_1 \in D_a$ for $t \leqslant \tau$. Thus appears the verification plan.

4. Let the points' masses $m_i = 1$, g_i be coordinate vectors, $g_i^{\cdot}$ be velocities. The points are attracted to a fixed center with a force inverse to the square of the distance to the center. The control forces $u_i^0 = u_{i,1} + u_{i,2} + u_{i,3}$. For $u_i^0 = 0$ the points move along an ellipses in the plane E. If the forces $u_i^0 \neq 0$, then they lie in this plane. By ψ_i we denote the polar angles of the points $l_i = (g_i, g_i^{\cdot})$. Let $z_i = z_i(l_i, \tau)$ be the first integrals of the linear equations of the perturbed motion closest to the ellipses of motion when $u_i = 0$, while $z_i(l_i, 0) = \psi_i$. Let $u_{i,1}^0$, $u_{i,2}^0$ be the projections of the forces onto the radius-vector $r_{\rho,i}$ and onto the transversal $\tau_{\rho,i}$. We take the equations of motion in the form $z_i = b_{i,1}(\tau)\,u_{i,1}^0 + b_{i,2}(\tau)\,u_{i,2}^0$. The control set is

$$U_b = \{u \mid u \in U(x),\ b_i^0(\tau) = (b_{i,1}(\tau),\ b_{i,2}(\tau)) \| u_{i,j} \text{ for } i = 1, 2,\ j = 1, 2, 3\}$$

The parallelism condition for $u_{i,j} \| b_i^0$ is mechanically trivial. As a result the controls $u_{i,j}$ can be made scalar and we can take the equations in the form

$$u_{i,j} \in \rho^{(1)},\ a_i(\tau) = |b_i^0(\tau)|,\ z_i^{\cdot} = u_i^0 a_i(\tau)$$

We shall examine two problems and two payoff functions $r_1(x)$, $r_2(x)$, $r_1(x) = |z|$; $z = (z_2 - z_1)$;

$$r_2(x) = (\min ||z| - 2\pi k| \text{ for } k = 1, 2, \ldots)$$

We commence with the problem having the payoff $r_1(x)$. The set $\xi_\alpha^0 = \{x \mid z \geqslant 0, x \overline{\in} \xi^0\}$, while the vectors

$$p_{z,i} \in \rho^{(1)},\ p_{\xi,i} \in \rho^{(1)},\ p_{\xi,i} \in [0, \mu_{i,3}]$$
$$\varphi_i = \{p_i \mid p_{z,i} = 1,\ p_{\mu,i,j}(-1)^i > 0,\ p_{\xi,i} \in [0, \mu_{i,3}],$$
$$|p_{\mu,i,1}| = 1\}$$
$$P = \varphi = \varphi_1 \times \varphi_2$$

Let us briefly describe the motions $y_b(y, t)$ for $e_{i,b} = 2\,|p_{\mu,i,2}|^{-1}$. At first we compute

$$\omega_{i,b} = f_{2,t}(\{a_i(t),\ \theta_\tau(x)\}),\ a_i^0(\tau) = \omega_{i,b,1}(x)$$
$$t_{i,\delta}(y) = t_{i,\delta}(\tau) = \inf \omega_{i,b,2}(\tau)$$
$$c_{i,j}(x, t) = \int_{\tau-t}^{\tau} a_i^j(s)\,ds \quad \text{for} \quad i, j = 1, 2$$
$$c_{i,3}(x, t) = 0 \text{ for } t \in [0, t_{i,\delta}(\tau))$$
$$c_{i,3}(x, t) = a_i^0(\tau) \text{ for } t \in [t_{i,\delta}(\tau), \tau]$$
$$a_{z,i}(y_b(y, t)) = \mu_{i,1} c_{i,1}(x, t) +$$
$$e_{i,b}(y)\,c_{i,2}(x, t) + p_{\xi,i}\,c_{i,3}(x, t)$$

For $x \in \xi_t(w_{1,\beta})$ the best motion $x_e(x, t)$ has the form

$$x_e(x,t) = a_x(y_b(y,t)) \text{ for}$$
$$p \in \zeta_{\alpha,2}(w_{1,\beta}) \in \rho^1$$
$$z_e(x,t) = a_z(x_e(x,t)) =$$
$$(a_z(y_b(y,t)) \text{ for } p_{\xi,i} = \mu_{i,3}$$

and for $e_{b,i} = e_{e,i}(x) = \sqrt{\mu_{i,2}/c_{i,2}(x,\tau)})$ when $x \in \xi$

We pass on to the analysis of the special sets F_i, $i = 1, 2, \ldots$

$$F_1 = \{x \mid x \in \xi_\alpha,\ \mu_{2,2} = \mu_{2,3} = 0\}$$

and we compute

$$h(x) = h_\alpha(y) \text{ for } p_{\xi,i} = \mu_{i,3},$$
$$e_{b,i} = e_{e,i}(x),\ i = 1, 2$$
$$h_0(x) = h_e(x) \text{ for } \mu_{1,2} = \mu_{1,3} = 0$$
$$F_{1,1} = F_1 \cap \{x \mid h_e(x) \geqslant 0\},$$
$$F_{1,2} = F_1 \cap \{x \mid h_e(x) < 0,\ h_0(x) \geqslant 0\}$$
$$F_{1,3} = \{x \mid h_0(x) < 0\} \cap F_1$$

In sets $F_{1,1}$, $F_{1,2}$ the set

$$\varphi_e(x) = \zeta_{\alpha,2}(w_{\beta,i}) \cap \{p \mid |p_{\mu,i,1}| = 1\}$$
$$\text{for } x \in F_{1,i}$$
$$\varphi_e(x) \in \rho^1 \text{ for } x \in F_1{}^0 = F_{1,1} \cup F_{1,2}$$
$$e_{i,e}(x) = 2\,|\,a_{p,\mu,i,2}(\varphi_e(x))\,|^{-1}$$

and the motion

$$x_e(x,t) = a_x\{y_b(y,t) \mid p \in \varphi_e(x)\}$$

has been defined. For $x \in F_{1,3}$ the motion $x_e(x,t)$ is unique; however, this follows from complicated computation rules and not from the essence of the problem. Furthermore, other motions with the same result exist since the function $\alpha_a(x)$ is independent of z when $x \in F_{1,3}$. By B_r we denote the whole set where $\partial\alpha_a(x)/\partial z = 0$.

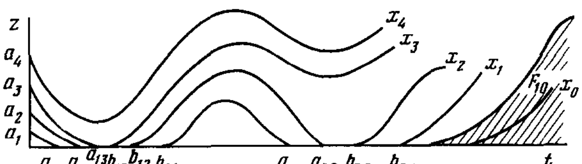

Fig.1

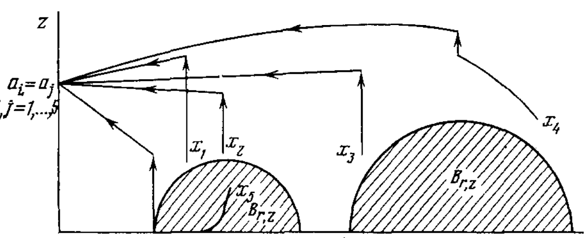

Fig.3

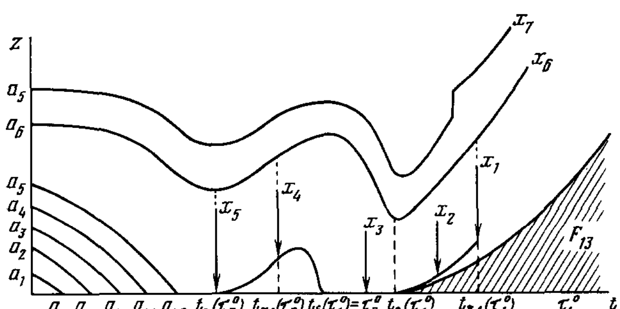

Fig.2

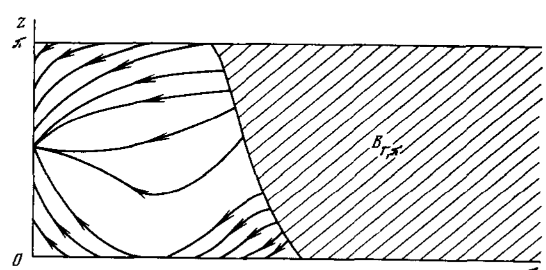

Fig.4

Figs.1—3 show the curves $z_e(x_j,t) = a_z(x_e(x,t))$. On Fig.1 (the case of $\mu_{1,3} = 0$) the function $e_{1,e}(x)$ is a zero, with respect to the variable $e_{1,b}$, of the function $h_\alpha(y)$. In the case at hand the minimum operation is not needed. The points $x_1, x_2 \in F_{1,3}$ of the motion $z_e(x_j,t)$ are parired by the tangency condition (they are not unique). For $x \in F_{1,3}$ the function $\alpha_a(x)$ is independent of z

$$\alpha_a(x) = \alpha_a(x_3) \text{ for } x_3 \in \{x \mid z = \rho_a(x, F_{1,2}) = 0\}$$

The sliding set $\theta_\omega = \{x \mid z = 0,\quad x \in F_{1,2},\quad y_e(x) = (\varphi_e(x), x) \in \theta_\lambda\}$ twice accepts the motions $z_e(x_j,t)$ for $x_j \in F_{1,2}$, $j = 1, 2$.

Fig.2 shows the case of $\mu_{1,2} = 0$. We denote $p_{3,e}(x) = a_{p,\xi,1}(\varphi_a(x))$; the set $\theta_\nu = \{x \mid t_{\delta,1}(\tau) - \tau = 0,\ h_e(x) < 0,\ x \in F_{1,2}\}$. When $x \in \theta_\nu$ the momentum $p_{3,e}(x)$ is zero of the function $h_\alpha(y)$ in the variable $p_{\xi,1}$. To the sliding set $\theta_\omega = \theta_\nu \cap \{x \mid p_{3,e}(x) = 0\}$ corresponds the motion $x_e(x$ $t)$ and the control

$$u_{1,3,e}(x) = \{u_{1,1} \mid \beta_z(x, u) = 0, \ u \in w_{,2}(y_e(x))\}$$

The set θ_ω borders on the two sets

$$\theta_{\sigma,1} = \{x \mid \tau - t_{\delta,1}(\tau) > 0\}$$
$$\theta_{\sigma,2} = \{x \mid t_\lambda(y_e(x)) = t(x) > 0\}$$

A transition with sliding onto the motion $z_e(x,t) > 0$ is accomplished on the boundaries $G_{\sigma,i} = fr\theta_i$ of sets $fr\theta_{\sigma,i} \cap \bar\theta_\omega$. We emphasize that the motion $z_e(x,t)$ can return again into set θ_ω. This return takes place smoothly (with tangency) through the boundary $G_{\sigma,2}$ and by an impulse through the boundary $G_{\sigma,1}$. The set $F_{1,\alpha} = \{x \mid x \in F_1, \ \mu_{1,2} > 0, \ \mu_{1,3} > 0\}$ is complex in that it contains the operation of minimum with respect to the variable $p_{\mu,1,2}$, but has no principal differences from the sets shown in Figs.1,2.

The set $F_2 = \{x \mid \mu_{1,2} = \mu_{1,3} = 0\}$ contains a dispersal surface /1/. We denote

$$\alpha_e^0(x, t) = (\alpha_{1,\xi}(x) \text{ for } z = 0, \ \tau = t)$$
$$w_r = f_{2,t}(\{\alpha_e^0(x, t), \ \theta_\tau\})$$
$$\alpha_r(x) = w_{r,1}(x), \quad \varkappa_r = \alpha_{1,\xi}(x) - \alpha_r(x)$$
$$t_r = \tau - \inf w_{r,2}$$

The result has the form

$$\alpha_a(x) = \alpha_{1,\xi}(x) \text{ for } x \in \xi_\zeta(w_{1,\beta}) =$$
$$\{x \mid \varkappa_r(x) \geqslant 0\} = A_r$$
$$\alpha_a(x) = \alpha_r(x) \text{ for } x \in B_r = F_2 \setminus A_r$$

The set $A_\sigma = \{x \mid z > 0, \ \varkappa_r(x) = 0\}$, $B_{r,1} = B_r \setminus A_\sigma$ is a dispersal surface. For $x \in B_{r,1}$ the controls $u_{i,2,e}(x) = u_{i,3,e}(x) = 0$ for $i = 1, 2$ and the motion turn about arbitrarily in the set $B_{r,1}$ up to the boundary. In Fig.3 the set $B_{r,1}$ is shaded (solution of the first problem).

We turn now to the second problem for the case when the required unperturbed motions of the points are circular orbits with a common center. We construct the vector $x_\xi(x)$ (of the permutation of the resources and the origin $0 \to \pi$)

$$a_z(x_\xi) = |\pi - |z||, \quad a_{\mu,i,j}(x_\xi) = \mu_{k,j}; \quad k \neq i, \quad i = 1, \ 2,$$
$$j = 1, \ 2, \ 3$$

We compute the function

$$\beta_{1,\xi}^0(x) = \min(\alpha_a(x), \ |\pi - \alpha_a(x_\xi(x))|)$$

and the function $\beta_{1,\xi} = \beta_{1,\xi}^0(x)$ for $|z| \leqslant \pi$, which has the period π in $|z|$. The function $\beta_{1,\xi}(x)$ is the solution of the second problem.

Typical motions of this problem are shown in Fig.4. The set $B_{r,\pi}$ is analogous to set B_r.

5. Suppose that a man $z_{2,0} = (\cos\theta, \sin\theta)$ can move with velocity $\theta^{\cdot} = u_2 \in P^{(1)}$ along a circle of unit radius with center $0'$ at the origin of a fixed system $Z_1^0, 0', Z_2^0$, while a lion $z_{1,0} \in P^{(2)}$, $z_{1,0} = u_1$ can move over the plane with velocity u_1; in addition

$$z_{2,0} = (\cos\theta, \sin\theta), \ z_1^0 = (z_{1,1}, z_{1,2}) = z_{2,0} - z_{1,0}$$

The problem is very similar to the well-known Littlewood problem. We take it that $\mu_{1,3} = \mu_{2,3} = 0$ and we write out the motions $x_e(\dot x, t)$ for $x \in \xi_\zeta \cdot (w_{1,\beta})$ in the variables $z^0 = (z_{1,0}, z_{2,0})$ and the variables $z_1^0 = (z_{1,1}, z_{1,2})$ in a moving system. We write out

$$n_i(x, t) = (\mu_{i,1} + \sqrt{\mu_{i,2}/\tau})\, t, \ n_i^0(x) = n_i(x, \tau)$$
$$z_n(x) = z_m(x) - z_{1,0} = (\cos(n_2^0(x)), \sin(n_2^0(x))) - z_{1,0}$$

$$j_n(x) = z_n(x)/|z_n(x)|, \ a_{z,2,0}(x_e) = (\cos(n_2(x, t)), \sin(n_2(x, t)))$$
$$a_{z,1,0}(x_e) = z_{1,0} + n_1(x, t)\, j_n(x)$$
$$z_{0,e} = (a_{z,1,0}(x), \ a_{z,2,0}(x_e))$$

The set $\xi_\zeta(w_{1,\beta})$ and the functions $h_a(y), h_e(x)$ play an analogous role in the construction. We start the analysis with details.

$$h_\gamma = \pi - \nu - |z_1|/q, \ q = \mu_{1,1} + \sqrt{\mu_{1,2}/\tau}, \ z = |z_{1,0}| \sin(\pi - \nu)$$
$$\pi - \nu = \arccos(z_{1,0} z_{2,0}) / |z_{1,0}|\,|z_{2,0}|$$
$$\Phi_1 = \{x \mid x \in \xi, \ \mu_{1,2} = \mu_{2,2} = 0\}, \ \Phi_{1,2} = \Phi_1 \cap \{x \mid h_e(x) \geqslant 0\}$$
$$\Phi_{1,2} = \{x \mid h(x) < 0, \ h_\gamma(x) \geqslant 0\}, \ \Phi_{1,3} = \Phi_1 \cap \{x \mid h_\gamma(x) < 0\}$$

The angle ν is counted counterclockwise from the vector z_1 up to the straight line Z_2' drawn from O' to the man. For $x \in \Phi_{1,1} \cup \Phi_{1,2}$ the motion $x_e(x, t)$ is unique. The motion $z_{0,e}(x, t)$ for $x_4 \in \Phi_{1,2}$ is shown in Fig.5.

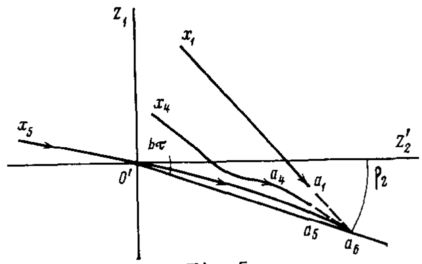

Fig.5

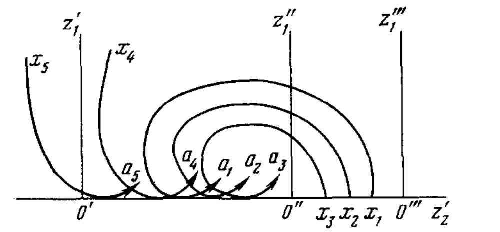

Fig.7

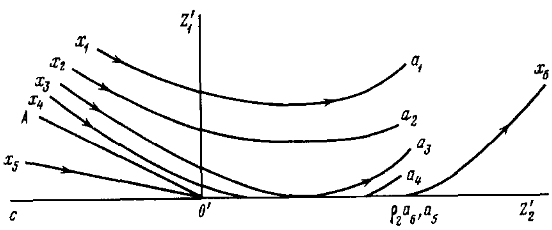

Fig.6

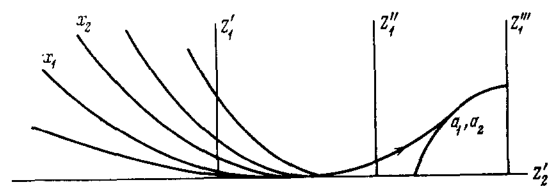

Fig.8

A deformation of the motion is connected with sliding over the set θ_ω. The lion's sliding is rectilinear in the moving axes, one of which is the axis Z_2', and curvilinear in the fixed system. The motions $z_{0,e}(x, t)$ for $x_1, x_2, x_3 \in \Phi_{1,1}$ have the lion's rectilinear trajectory. The motion $z_{0,e}(x, t)$ for $x_5 \in \Phi_{1,3}$ has the lion's rectilinear absolute trajectory directed toward the point O' until $z_{0,e}(x, t) \in \Phi_{1,3}$. This segment a_5, O' is shown in Fig.5. Fig.6 corresponds to motions in the moving axes Z_1', O', Z_2'. The set

$$\Phi_2 = \{x \mid \mu_{2,2} = 0,\ x \in \xi\}$$

The motion $x_e(x, t)$ for

$$x \in \Phi_{2,1} = \{x \mid h_e(x) \geqslant 0\} \cap \Phi_2$$

have been described at the beginning of the example. When

$$x \in \Phi_{2,2} = \{x \mid h_e(x) < 0\} \cap \Phi_2$$

the typical motions shown in Fig.7 appear. The point O'' is the point of absolute minimum with respect to z^0 of the function $\alpha_{1,\xi}(x)$. Then the points x_1, x_2, x_3 lie to the right of point O''. The motion $z_e(x, t)$ in the moving system Z_1', O', Z_2' leaves the symmetry axis Z_2' and returns to it again.

The set

$$\Phi_3 = \{x \mid \mu_{1,1} = \mu_{2,1} = 0\}$$

picks out the set

$$\xi_3 = \{x \mid z = 0,\ x \in \Phi_3\}$$

The set

$$\gamma_3 = \{y \mid x \in \xi_3,\ p_{\mu,2,2} = \infty,\ |p_{z,1}| = 1,\ p_{z,1} \,||\, Z_2'\}$$

corresponds to the second player's zero control $u_{2,2} = 0$. For $y \in \gamma_3$ the motion $x_b(y, t) = a_x(y_b(y, t))$, the symmetry axis is fixed. We compute the operators

$$\zeta_\theta = f_{2,t}(\{\alpha_{1,\xi}(x_b(y, t)),\ \theta_\tau(x)\})$$
$$\zeta_\delta = f_{2,1}(\{\zeta_{\theta,1}(y),\ \{p \mid p \in \gamma_3\}\})$$
$$\alpha_{1,\gamma}(x) = \zeta_{\delta,1}(x),\ \tau_{1,\gamma}(x) = \tau - (\inf \zeta_{\theta,1}(y) \text{ for } p \in \zeta_{\sigma,2}(x))$$

and we construct the sets and the functions

$$\xi_4 = \xi_\zeta(w_{1,\beta}) \cup \xi_3;\ \alpha_\sigma(x) = \alpha_{1,\xi}(x) \text{ for } x \in \xi_\zeta(w_{1,\beta})$$
$$\alpha_\sigma(x) = \alpha_{1,\gamma}(x) \text{ for } x \in \xi_3,\ \xi_5 = \xi \setminus \xi_4$$
$$w_{3,\beta} = \{\alpha_\sigma(x),\ y_b(y, t),\ \gamma^0(\xi_5)\}$$
$$\alpha_a(x) = \alpha_{1,\xi}(x) \text{ for } x \in \xi_\zeta(w_{1,\beta})$$
$$\alpha_a(x) = \alpha(w_{3,\beta}) \text{ for } x \in \Phi_3 \setminus \xi_\zeta(w_{1,\beta})$$

The formulas written out give the answer when $x \in \Phi_3$. The motions are shown in Fig.8.

REFERENCES

1. ISAACS R.P., Differential Games. New York, John Wiley and Sons, Inc., 1965.
2. KRASOVSKII N.N. and SUBBOTIN A.I., Positional Differential Games. Moscow, NAUKA, 1974.

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